About the HELM Project

HELM (Helping Engineers Learn Mathematics) materials were the outcome of a three-year curriculum development project undertaken by a consortium of five English universities led by Loughborough University, funded by the Higher Education Funding Council for England under the Fund for the Development of Teaching and Learning for the period October 2002 September 2005.

HELM aims to enhance the mathematical education of engineering undergraduates through a range of flexible learning resources in the form of Workbooks and web-delivered interactive segments.

HELM supports two CAA regimes: an integrated web-delivered implementation and a CD-based version.

HELM learning resources have been produced primarily by teams of writers at six universities:
Hull, Loughborough, Manchester, Newcastle, Reading, Sunderland.

HELM gratefully acknowledges the valuable support of colleagues at the following universities and colleges involved in the critical reading, trialling, enhancement and revision of the learning materials: Aston, Bournemouth & Poole College, Cambridge, City, Glamorgan, Glasgow, Glasgow Caledonian, Glenrothes Institute of Applied Technology, Harper Adams University College, Hertfordshire, Leicester, Liverpool, London Metropolitan, Moray College, Northumbria, Nottingham, Nottingham Trent, Oxford Brookes, Plymouth, Portsmouth, Queens Belfast, Robert Gordon, Royal Forest of Dean College, Salford, Sligo Institute of Technology, Southampton, Southampton Institute, Surrey, Teesside, Ulster, University of Wales Institute Cardiff, West Kingsway College (London), West Notts College.

HELM Contacts:

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HELM Workbooks List

1	Basic Algebra	26	Functions of a Complex Variable
2	Basic Functions	27	Multiple Integration
3	Equations, Inequalities & Partial Fractions	28	Differential Vector Calculus
4	Trigonometry	29	Integral Vector Calculus
5	Functions and Modelling	30	Introduction to Numerical Methods
6	Exponential and Logarithmic Functions	31	Numerical Methods of Approximation
7	Matrices	32	Numerical Initial Value Problems
8	Matrix Solution of Equations	33	Numerical Boundary Value Problems
9	Vectors	34	Modelling Motion
10	Complex Numbers	35	Sets and Probability
11	Differentiation	36	Descriptive Statistics
12	Applications of Differentiation	37	Discrete Probability Distributions
13	Integration	38	Continuous Probability Distributions
14	Applications of Integration 1	39	The Normal Distribution
15	Applications of Integration 2	40	Sampling Distributions and Estimation
16	Sequences and Series	41	Hypothesis Testing
17	Conics and Polar Coordinates	42	Goodness of Fit and Contingency Tables
18	Functions of Several Variables	43	

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Introduction

49.1

1. Background to the HELM project

In 1997, funding was made available by Loughborough University for the ‘Open Learning Project’ in

3. HELM project Workbooks

50 Workbooks are available which comprise:

- 46 Student Workbooks written specifically with the typical engineering student in mind containing mathematical and statistical topics, worked examples, tasks and related engineering examples.
- A Workbook containing supplementary mathematical topics and physics case studies.
- A Workbook containing Engineering Case Studies ranging over many engineering disciplines.
- A Students' Guide (this document)
- A Tutor's Guide

The main project materials are the Workbooks which are subdivided into manageable Sections. As far as possible, each Section is designed to be a self-contained piece of work that can be attempted by the student in a few hours. In general, a whole Workbook typically represents 2 to 3 weeks' work. Each Workbook Section begins with statements of prerequisites and the desired learning outcomes.

The Workbooks include (a) worked examples, (b) tasks for students to undertake with space for students to attempt the questions, and, often, intermediate results provided to guide them through problems in stages, and (c) exercises where normally only the answer is given.

4. HELM project Interactive Learning Resources

The project has 80 Interactive Learning Resources, which link to most of the lower level Mathematics and Statistics Workbooks. These enable web-based versions of the Workbooks to contain some audio and some simple animations. Revision exercises with randomly generated questions are provided for the benefit of students working independently.

5. HELM project Assessment Regime

The HELM assessment strategy is based on using Computer-Aided Assessment (CAA) to encourage self-assessment to verify that the appropriate skills have been learned. The project's philosophy is that assessment should be at the heart of any learning and teaching strategy and Loughborough University's own implementation of HELM makes extensive use of CAA to support the students' learning.

HELM provides an integrated web-delivered CAA regime for both self-testing and formal assessment, with around 5000 questions, most have a page of specific feedback.

6. HELM Consortium and Triallist Institutions and Individual Contributors

HELM learning resources have been produced primarily by a consortium of writers and developers at five universities:

Hull, Loughborough, Manchester, Reading, Sunderland.

The HELM consortium gratefully acknowledges the valuable support of many colleagues at their own institutions and at the following institutions involved in additional writing, critical reading, trialling and revising of the learning materials.

Universities
Aston
Cambridge
City
Glamorgan
Glasgow
Glasgow Caledonian
Hertfordshire
Leicester
Liverpool
London Metropolitan
Newcastle
Northumbria
Nottingham
Nottingham Trent
Oxford Brookes
Plymouth
Queen's Belfast
Robert Gordon
Southampton
Southampton Solent
Surrey
Teesside
Ulster
University of Wales Institute Cardiff

HELM Workbooks

49.2

1. List of Workbooks

No.	Title	Pages
1	Basic Algebra	89
2	Basic Functions	75
3	Equations, Inequalities & Partial Fractions	71
4	Trigonometry	77
5	Functions and Modelling	49
6	Exponential and Logarithmic Functions	73
7	Matrices	50
8	Matrix Solution of Equations	32
9		

No.	Title	Pages
37	Discrete Probability Distributions	60
38	Continuous Probability Distributions	27
39	The Normal Distribution	39
40	Sampling Distributions and Estimation	22
41	Hypothesis Testing	42
42	Goodness of Fit and Contingency Tables	24
43	Regression and Correlation	32
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45	Non-parametric Statistics	36
46	Reliability and Quality Control	38
47	Mathematics and Physics Miscellany	70
48	Engineering Case Studies	97
49	Student's Guide	57
50	Tutor's Guide	143

2. Nomenclature used for problems

- **Examples** are problems with fully worked solutions.
- **Engineering Examples** (found in most Mathematics Workbooks but not the Statistics Workbooks) are problems with an engineering context having fully worked solutions.
- **Tasks** are problems with spaces for the student's working, followed by fully worked solutions. Many Tasks are often broken up into stages with the answer to a stage given before the next stage is reached. [Note: Some tutors may provide workbooks without these worked solutions.]
- **Exercises** are problems for the student to do without spaces provided for the student's working. In general they do not have fully worked solutions, merely answers, but exceptions are: Numerical Workbooks 30-33 and Statistics Workbooks 35-46 which do have fully worked solutions.

3. Notation used

In general HELM uses italic serif font letters (e.g. $f(x)$) to represent functions, variables and constants. However, as exceptions HELM Workbooks use the following non-italic sans-serif letters:

Mathematics

e for the exponential constant and for the exponential function (primarily use in introductory Workbook 6, elsewhere e is often used)

i where $\mathrm{i}^2 = -1$

ln for natural logarithm

Statistics

E for Expectation

P for Probability

V for Variance

M for Median

Complex numbers

HELM uses i rather than j to represent $\sqrt{(-1)}$ so $i^2 = -1$, although there are one or two exceptions to this (in Workbook 48: Engineering Case Studies).

Vectors

HELM uses underlining of vectors rather than using bold e.g. $\underline{a}$

HELM uses $\underline{\hat{n}}$ for the unit normal vector but does not put the ˆ on the basic unit vectors in the x, y and z directions which have the standard symbols

Description of HELM Workbook layout

Complex Arithmetic 10.1

Introduction

Complex numbers are used in many areas of engineering an

[redacted]

Key Points.
Take special note of these.

Key Point 1

The symbol i is such that

...ou to try with space for your working.
... presented after solution box.

Task for y...
Answe...

Task

Answer

$z + 2w = 4 + 5i$

Your solution

(c)

Your solution

Worked example.
Solution with explanation follows in box..

Example 2

Find $z + w$ if $z = 2 - 3\mathrm{i}$ and $w = 2 + \mathrm{i}$

HELM Electronic Learning Resources 49.3

1. Introduction

HELM has 50 Workbooks and 80 Interactive Learning Resources and linked Revision Questions (with inbuilt randomisation).

The Interactive Learning Resources provide web-based lessons to match some Sections of many of the more elementary Workbooks and contain animations and interactivity to generate interest and have linked Revision Exercises where randomly generated questions are provided for the benefit of students working independently.

These Interactive Learning Resources and linked Revision Exercises have been found to be especially useful for supporting students who find it difficult to cope with the mathematical demands of their programmes.

The animations are also useful for illustrating lectures and for revision.

The tutor will provide guidance as to how the materials are to be used.

2. Accessing HELM electronic learning resources

The web based versions of HELM Interactive Learning Resources can be accessed via

http://helm.lboro.ac.uk/cal/

or via any specific web address (url) given to you by your tutor.

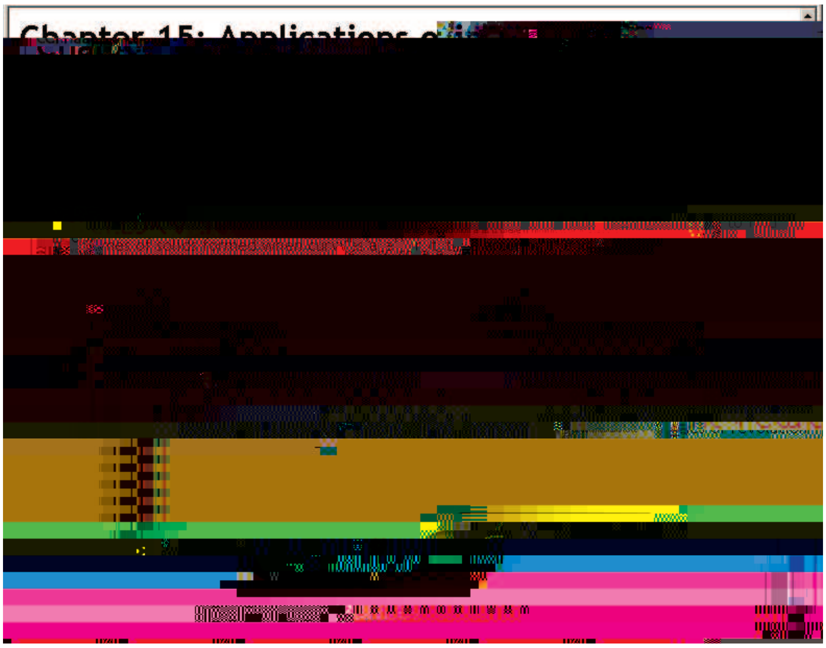
Once you access this web page, you will see four links as shown below:

Clicking on either an icon or the hypertext link below the icon will take you to the corresponding web page containing links to the selected learning resources.

Workbooks:

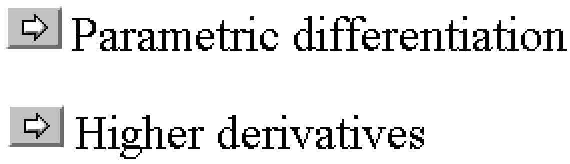
There are fifty HELM Workbooks available to HEFCE-funded Higher Education Institutions in England and Northern Ireland from the Mathematics Education Centre at Loughborough University.

Access to these Workbooks is restricted to staff and students of these institutions and is controlled by each institution.



The introduction page presents you with the title of the Workbook and the section that is covered

As you see, there may be one or more buttons that will take you to a particular subsection of the lesson. For example, referring to the figure given below, clicking on the top button would take you to a section on Parametric differentiation; clicking the bottom button would take you to a section on Higher Derivatives.



After you click on one of the buttons, the front page will disappear and you will see the first page of your chosen section. Look at the top right-hand corner of the screen: inside a box you will see in **green**

So that you have a permanent record, the icon functions are also described below.

To move to the next page of a section, click on: 

To move to the previous page of a section, click on: 

To return to the front page, click on: 

To quit00cmq1001orq10nns2e,k on:

- + and t
- 0

BODMAS: (Brackets, 'Of', Division, Multiplication, Addition, Subtraction):

1. **B**rackets take highest priority - deal with items inside a pair of brackets first.
2. **O**f is a form of multiplication (e.g. 'half of 10' means $1/2 \times 10$) and comes next.
3. **D**ivision and **M**ultiplication come next and left-to-right order is required (e.g. $4 \div 7x \times k$ is evaluated as $(4 \div 7) \times k$ and not as $4 \div (7x \times k)$).
4. **A**ddition and Subtraction come last (in either order will do but left-to-right is normal).

When faced with several operations at the same level of precedence the left-to-right order is normally used, but it is not essential.

Beware of calculators

Not all calculators follow these conventions in all circumstances, and ambiguities can arise, so you should check what you get for operations such as $4 \div 7 \times 7$, $2 - 3^2$ and 3^{2+1}. Inserting brackets will sort out these problems if you are unsure what your calculator will do, or if you want to force it to do something it won't do otherwise.

7. Equality and Identity

The equals sign (=) is often wrongly used as a shorthand symbol for "*gives*" or "*leads to*" or like phrases. For instance, when finding the third derivative of $x^3 + 2x - 3$, some students will write

$$\frac{d^3}{dx^3}(x^3 + 2x - 3) = 3x^2 + 2 = 6x = 6$$

These four expressions are not equal of course.

This practice is more annoying to the tutor than harmful to the student!

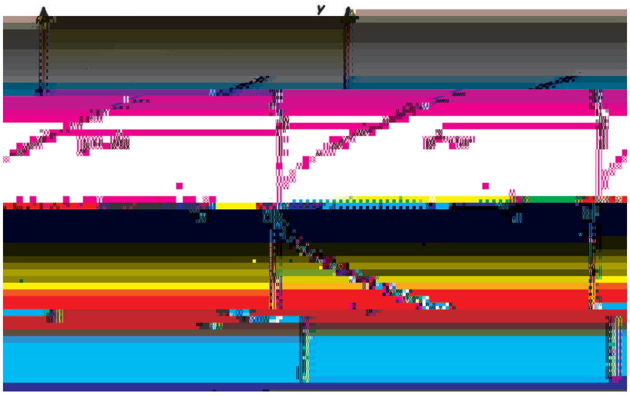
The use of = is commonplace throughout mathematics and hides the distinction between expressions which are true for particular values (e.g. $2x = 2$) and those, which are ALWAYS true (e.g. $2x = x + x$). The special identity symbol ($\equiv$) is (or rather can be) used for these: e.g. $2x \equiv x + x$. This symbol has been used sometimes in the HELM Workbooks where emphasis is important (especially in Workbook 1: Basic Algebra and in Workbook 4: Trigonometry) but we *have not done so consistently* - it just isn't the way mathematicians and engineers work! In practice it is nearly always obvious from the context, which is meant.

8. Notational problems

Square root symbol

Every positive number has two real square roots. The expression $\sqrt{n}$ actually means "the **non-negative** square root of n," but many think it can represent either of the square roots of n - i.e., it represents two numbers. This error is actually encouraged by the common practice of referring to $\sqrt{n}$ as "**the** square root of " instead of the more carefully worded "the **positive** square root of ". In fact even that phrase isn't quite correct in all circumstances since it could be zero!

The graphs of $y = \sqrt{x}$ and $y^2 = x$ below illustrate the point:



If you want to refer to both roots then you must use $\pm\sqrt{\ }$, as in the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What about $x^{\frac{1}{2}}$? Usually this is taken to mean $\sqrt{x}$ but, particularly in complex number work, it can mean **any** value of the root. So $y = x^{\frac{1}{2}}$ could mean either of the graphs shown above!

Another common error is to replace $\sqrt{1 - \sin^2\theta}$ by $\cos\theta$ (because $1 - \sin^2\theta \equiv \cos^2\theta$). This is wrong because $\cos\theta$ can be negative whereas $\sqrt{\ }$ is never negative, so the result should be expressed as $|\cos\theta|$.

Trigonometric inverses

The expression $\sin^k x$ is interpreted in different ways, depending on the value of k.

$$\sin^3 x \equiv (\sin x)^3 \quad \text{and similarly for cos, tan, sec, cosec and cot}$$

but

$\sin^{-1} x$ means the inverse sine function, sometimes written as arcsin (x), and similarly for cos, tan, sec, cosec and cot.

Note that $\arcsin(x) = (\sin x)^{-1}$ but $\operatorname{cosec}(x)$

Irreversible operations

Some operations are not reversible, and using them can introduce new solutions (called extraneous solutions) not valid for the original equation.

1. Square rooting is irreversible: e.g. $x = -9$ has only one solution, which is $x = -9$ of course, but after squaring both sides we get $x^2 = 81$, which has two solutions, $x = 9$ and $x = -9$.

2. Mul8.998cmBT/F4711.9554.1310Td.TJ/F471351(s)-1(o)1(l)-1(utions)-11.955Tf3/F4711.909090909090

3. Taking square root: e.g. $\sqrt{x^2 + y^2} = \sqrt{x^2} + \sqrt{y^2}$
4. Exponentiation: $\exp(x + y) = \exp(x) + \exp(y)$
5. Taking logarithm: $\log(x + y) = \log(x) + \log(y)$
6. Matrices (inversion): $(A + B)^{-1} = A^{-1} + B^{-1}$

This is a common mistake made by first year undergraduates who have not studied mathematics for some time.

12. Commutativity of operations

Two operations f and g **commute** if you get the same result when you perform them in either order: i.e. $f(g(x)) = g(f(x))$. E.g. if f means “doubling” and g means “trebling” then $f(g(5)) = f(15) = 30$ and $g(f(5)) = g(10) = 30$ so $f(g(5)) = g(f(5))$.

This is **true for some** combinations of operations. Examples are:

1. Powers and roots of positive numbers: $(\sqrt{x})^3 = \sqrt{(x^3)}$
2. Multiplication by a constant and integration: $2\int u\,dx = \int 2u\,dx$

It is **not true for most** combinations of operations. Examples are:

1. “Doubling” and “Adding 1” = “Adding 1” and “Doubling”
2. Powers and addition: $(x + 1)^3 = x^3 + 1^3$
3. Taking cosine and squaring: $\cos(x^2) = \{\cos(x)\}^2$
4. Multiplication and differentiation: $(u \times v) = u \times v$
5. Division and integration: $\int (u/v)\,dx = \int u\,dx / \int v\,dx$

13. Dimensions and scaling

Dimensional analysis is an important topic for engineers and is treated in Workbook 47. It doesn’t tell you if you have the right formula or answer, but it can indicate that something must be wrong. Here are some simple examples:

1. If you’re asked to find a length, and your answer is some number of square cms, then you must have made an error somewhere.

2. If you're asked to find an area and your answer is a negative number, then you know you've made an error somewhere UNLESS it is a calculus problem (where an area below the axis may be represented as a negative quantity).

3. The formula for the area, S, of a triangle with sides a, b, c must have dimensions of area so it cannot possibly be either of the following:

   $S = a \times b \times c$ or $S = a + b + c$

   It might in theory be

   $S = (a + b + c)^2$

   which has the right dimensions for area, though that isn't actually correct of course!

   There is in fact a complicated formula involving only a, b, c for S, called Heron's formula:

   $S = \sqrt{\{(a + b + c)(b + c - a)(c + a - b)(a + b - c)/16\}}$.

   You can check that this is dimensionally correct.

Unit Conversion

A related problem is converting from one unit to another. Just because 1 m = 100 cm does not mean that 1 m^3 = 100 cm^3. Obvious, perhaps, but an easy mistake to make when not concentrating. In fact, of course, there are three dimensions here so the scale factor is 100^3 and 1 m^3 = 1 000 000 cm^3.

Scaling error

If the question is a real-world problem, you should ask: "Is my answer sensible?" For instance, if you are given a list of the main components used in the manufacture of a truck and are asked to estimate its unladen weight, and you come up with an answer of 1000 tonnes, then you must have made a mistake either in the calculations or in the units.

14. Some further traps

It is important to remember the following:

(a) Cancelling in fractions

Don't fall into the trap of *partial cancelling*.

This is correct:

$$\frac{(x-1)\cancel{(x+2)}}{(x+3)\cancel{(x+2)}} = \frac{(x-1)}{(x+3)} \quad \text{(pro ided } x \neq -2)$$

but this is NOT correct:

$$\frac{(x-1)+\cancel{(x+2)}}{(x+3)\cancel{(x+2)}} = \frac{(x-1)+1}{(x+3)}$$

You only cancel once when the factors in the numerator are multiplied but you must cancel each

Of the following three statements only the first two are known with any certainty by most students:

Given a twice differentiable function f for which $f\ (a) = 0$

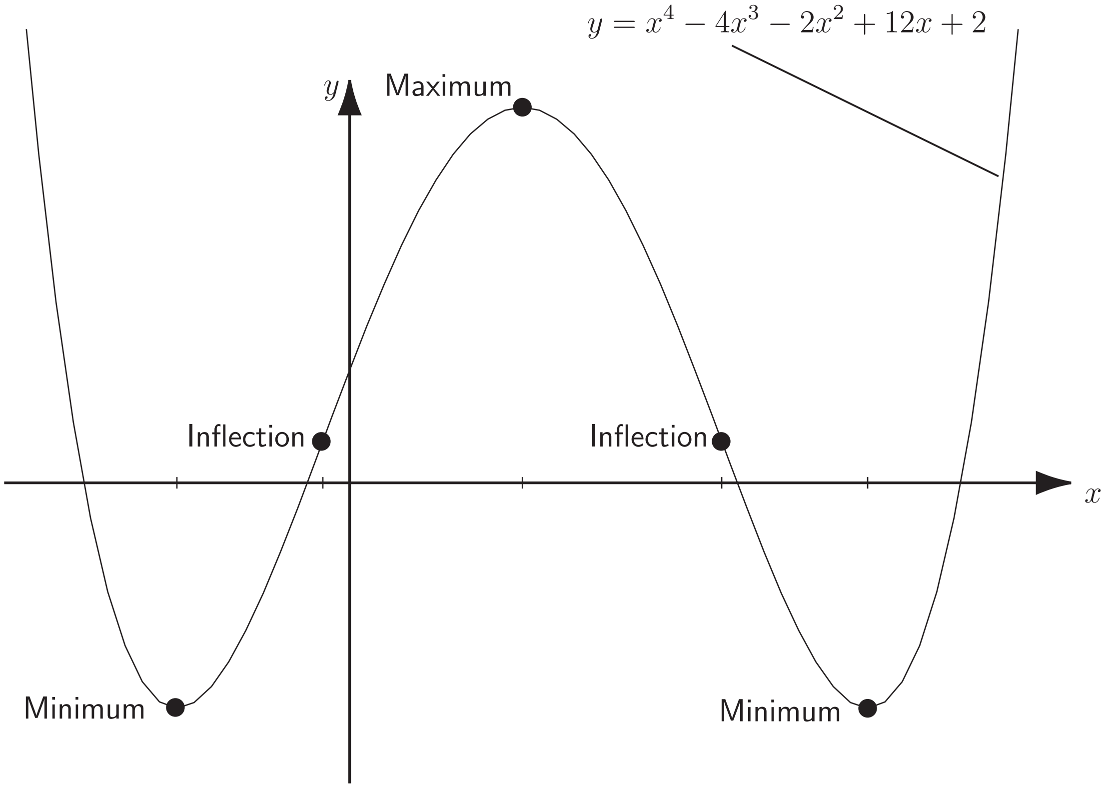
(1) If $f\ (a) > 0$, then $f(x)$ has a minimum when $x = a$,

(2) If $f\ (a) < 0$, then $f(x)$ has a maximum when $x = a$,

(3) If $f\ (a) = 0$, then $f(x)$ has minimum or a maximum or a point of inflection when $x = a$.

Many students think (3) **always** leads to a point of inflection but the graph of $f(x) = x^4$ clearly shows this to be untrue when $x = 0$.

Another misconception is that a point of inflection **requires** $f\ (a) = 0$. This is not true as can easily be seen, for example, on the sine curve. This raises another point - for any continuous function there is always a point of inflection between every local minimum and local maximum. The graph below highlights these features.



Maxima and Minima without Calculus

Students all too readily turn to the calculus when needing to find maxima and minima. There are,

Example 1

Find the minimum value of $f(x) \equiv x^2 + 2x + 3$.

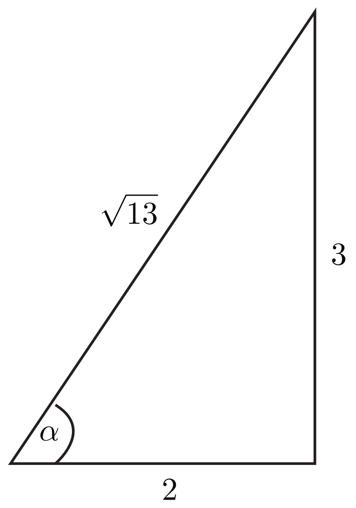
Completing the square gives $f(x) = (x+1)^2 + 2$.

This clearly is a minimum when $x = -1$ and there $f(x)$ has value 2.

Example 2

Find the maximum value of $f(x) \equiv 2\sin(x) + 3\cos(x)$.

Using the trigonometric identity $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$ and utilising the triangle in the diagram we have



$$\begin{aligned} f(x) &= \sqrt{13}\left(\frac{2}{\sqrt{13}}\sin x + \frac{3}{\sqrt{13}}\cos x\right) \\ &= \sqrt{13}[\cos\alpha \sin x + \sin\alpha \cos x] \\ &= \sqrt{13}\sin(x+\alpha) \end{aligned}$$

This clearly has a maximum value of $\sqrt{13}$ at $x = \dfrac{\pi}{2} - \alpha$ (for example), which is where $\sin(x+\alpha) = \sin\left(\dfrac{\pi}{2}\right) = 1$.

Some Useful Websites 49.5

(a) The reader is referred to the excellent website

http://atlas.math.vanderbilt.edu/ schectex/commerrs/

in which useful discussion of many intriguing errors and pitfalls are found and tips on avoiding them.

In particular, errors in calculus including integration by parts and solving differential equations are discussed at some length.

(b) See also the *Maths Mistakes* website (a site dedicated to the listing of mathematical mistakes made by advertisers, the media, reporters, politicians, activists and others) where you can marvel at the mistakes which others (not students) make:

http://members.cox.net/mathmistakes/

(c) Another valuable site is Eric Weisstein's Mathworld supported by Wolfram Research:

http://mathworld.wolfram.com/

which has a wealth of material, where you can look up definitions and formulae etc(l)-1(cul)-1(us)-321a8

Workbook 6 - Exponential and Logarithmic Functions (73 pages)	
6.1	The Exponential Function
6.2	The Hyperbolic Functions
6.3	Logarithms
6.4	The Logarithmic Function
6.5	Modelling Exercises
6.6	Log-linear Graphs
Workbook 7 - Matrices (50 pages)	
7.1	Introduction to Matrices
7.2	Matrix Multiplication
7.3	Determinants
7.4	The Inverse of a Matrix
Workbook 8 - Matrix Solution of Equations (32 pages)	
8.1	Solution by Cramer's Rule
8.2	Solution by Inverse Matrix Method
8.3	Solution by Gauss Elimination
Workbook 9 - Vectors (66 pages)	
9.1	Basic Concepts of Vectors
9.2	Cartesian Components of Vectors
9.3	The Scalar Product
9.4	The Vector Product
9.5	Lines and Planes
Workbook 10 - Complex Numbers (34 pages)	
10.1	Complex Arithmetic
10.2	Argand Diagrams and the Polar Form
10.3	The Exponential Form of a Complex Number
10.4	De Moivre's Theorem
Workbook 11 - Differentiation (58 pages)	
11.1	Introducing Differentiation
11.2	Using a Table of Derivatives
11.3	Higher Derivatives
11.4	Differentiating Products and Quotients
11.5	The Chain Rule
11.6	Parametric Differentiation
11.7	Implicit Differentiation

Workbook 12 - Applications of Differentiation (63 pages)	
12.1	Tangents and Normals
12.2	Maxima and Minima
12.3	The Newton-Raphson Method
12.4	Curvature
12.5	Differentiation of Vectors
12.6	Case Study: Complex Impedance
Workbook 13 - Integration (62 pages)	
13.1	Basic Concepts of Integration
13.2	Definite Integrals
13.3	The Area Bounded by a Curve
13.4	Integration by Parts
13.5	Integration by Substitution and Using Partial Fractions
13.6	Integration of Trigonometric Functions
Workbook 14 - Applications of Integration 1 (34 pages)	
14.1	Integration as the Limit of a Sum
14.2	The Mean Value and the Root-Mean-Square Value
14.3	Volumes of Revolution
14.4	Lengths of Curves and Surfaces of Revolution
Workbook 15 - Applications of Integration 2 (31 pages)	
15.1	Integration of Vectors
15.2	Calculating Centres of Mass
15.3	Moment of Inertia
Workbook 16 - Sequences and Series (51 pages)	
16.1	Sequences and Series
16.2	Infinite Series
16.3	The Binomial Series
16.4	Power Series
16.5	Maclaurin and Taylor Series
Workbook 17 - Conics and Polar Coordinates (43 pages)	
17.1	Conic Sections
17.2	

Workbook 19 - Differential Equations (70 pages)	
19.1	Modelling with Differential Equations
19.2	First Order Differential Equations
19.3	Second Order Differential Equations
19.4	Applications of Differential Equations
Workbook 20 - Laplace Transforms (73 pages)	
20.1	Causal Functions
20.2	The Transform and its Inverse
20.3	Further Laplace Transforms
20.4	Solving Differential Equations
20.5	The Convolution Theorem
20.6	Transfer Functions
Workbook 21 z-Transforms (96 pages)	
21.1	z-Transforms
21.2	Basics of z-Transform Theory
21.3	z-Transforms and Difference Equations
21.4	Engineering Applications of z-Transforms
21.5	Sampled Functions
Workbook 22 - Eigenvalues and Eigenvectors (53 pages)	
22.1	Basic Concepts
22.2	Applications of Eigenvalues and Eigenvectors
22.3	Repeated Eigenvalues and Symmetric Matrices
22.4	Numerical Determination of Eigenvalues and Eigenvectors
Workbook 23 - Fourier Series (73 pages)	
23.1	Periodic Functions
23.2	Representing Periodic Functions by Fourier Series
23.3	Even and Odd Functions
23.4	Convergence
23.5	Half-range Series
23.6	The Complex Form
23.7	An Application of Fourier Series
Workbook 24 - Fourier Transforms (37 pages)	
24.1	The Fourier Transform
24.2	Properties of the Fourier Transform
24.3	Some Special Fourier Transform Pairs

Workbook 25 - Partial Differential Equations (42 pages)	
25.1	Partial Differential Equations
25.2	Applications of PDEs
25.3	Solution using Separation of Variables
25.4	Solutions using Fourier Series
Workbook 26 - Functions of a Complex Variable (58 pages)	
26.1	Complex Functions
26.2	Cauchy-Riemann Equations and Conformal Mappings
26.3	Standard Complex Functions
26.4	Basic Complex Integration
26.5	Cauchy’s Theorem
26.6	Singularities and Residues
Workbook 27 - Multiple Integration (83 pages)	
27.1	Introduction to Surface Integrals
27.2	Multiple Integrals over Non-rectangular Regions
27.3	Volume Integrals
27.4	Changing Coordinates
Workbook 28 - Differential Vector Calculus (53 pages)	
28.1	Background to Vector Calculus
28.2	Differential Vector Calculus
28.3	Orthogonal Curvilinear Coordinates
Workbook 29 - Integral Vector Calculus (77 pages)	
29.1	Line Integrals Involving Vectors
29.2	Surface and Volume Integrals
29.3	Integral Vector Theorems
Workbook 30 - Introduction to Numerical Methods (64 pages)	
30.1	Rounding Error and Conditioning
30.2	Gaussian Elimination
30.3	LU Decomposition
30.4	Matrix Norms
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Mathematics Facts and Formulae

49.8

On the following pages are collections of useful Facts and Formulae. They were developed by Tony Croft and Geoff Simpson and are reproduced with the permission of Loughborough University Mathematics Education Centre.

Mathematical Topics

Algebra

Trigonometry

The Sine Rule and Cosine Rule

Hyperbolic Functions

Differentiation

Integration

Complex Numbers

Vectors

Sequences and Series

Matrices and Determinants

The Binomial Coefficients

Graphs of Common Functions

The Greek Alphabet

Algebra

$$(x + k)(x - k) = x^2 - k^2$$

$$(x + k)^2 = x^2 + 2kx + k^2, \quad (x - k)^2 = x^2 - 2kx + k^2$$

$$x^3 \pm k^3 = (x \pm k)(x^2 \quad kx + k^2)$$

Formula for solving a quadratic equation:

$$\text{if } ax^2 + bx + c = 0 \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Laws of Indices

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{mn}$$

$$a^0 = 1 \qquad a^{-m} = \frac{1}{a^m} \qquad a^{1/n} = \sqrt[n]{a} \qquad a^{\frac{m}{n}} = (\sqrt[n]{a})^m$$

Laws of Logarithms

For any positive base b (with $b = 1$)

$$\log_b A = c \quad \text{means} \quad A = b^c$$

$$\log_b A + \log_b B = \log_b AB, \qquad \log_b A - \log_b B = \log_b \frac{A}{B},$$

$$n \log_b A = \log_b A^n, \qquad \log_b 1 = 0, \qquad \log_b b = 1$$

Formula for change of base: $\log_a x = \frac{\log_b x}{\log_b a}$

Logarithms to base e, denoted $\log_e$ or alternatively ln are called *natural logarithms*. The letter e stands for the exponential constant which is approximately 2.718.

Partial fractions

For *proper fractions* $\frac{P(x)}{Q(x)}$ where P and Q are polynomials with the degree of P less than the degree of Q:

a *linear factor* $ax + b$ in the denominator produces a partial fraction of the form $\frac{A}{ax+b}$

repeated linear factors $(ax + b)^2$ in the denominator produce partial fractions of the form $\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$

a *quadratic factor* $ax^2 + bx + c$ in the denominator produces a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$

Improper fractions require an additional term which is a polynomial of degree $n - d$ where n is the degree of the numerator and d is the degree of the denominator.

Inequalities:

$a > b$ means a is greater than b

$a < b$ means a is less than b

$a \geqslant b$ means a is greater than or equal to b

$a \leqslant b$ means a is less than or equal to b

Trigonometry

Degrees and radians

$$360^\circ = 2\pi \text{ radians}, \quad 1^\circ = \frac{2\pi}{360} = \frac{\pi}{180} \text{ radians}$$

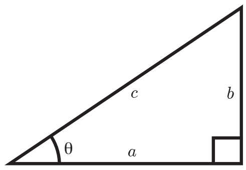
$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \approx 57.3^\circ$$

Trig ratios for an acute angle θ:

$$\sin\theta = \frac{\text{side opposite to } \theta}{\text{hypotenuse}} = \frac{b}{c}$$

$$\cos\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{a}{c}$$

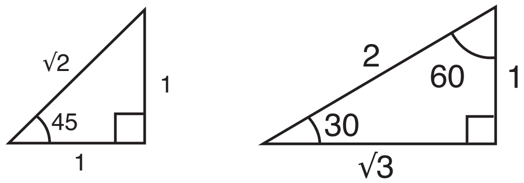
$$\tan\theta = \frac{\text{side opposite to } \theta}{\text{side adjacent to } \theta} = \frac{b}{a}$$



Pythagoras' theorem

$$a^2 + b^2 = c^2$$

Standard triangles:



$$\sin 45^\circ = \frac{1}{\sqrt{2}}, \qquad \cos 45^\circ = \frac{1}{\sqrt{2}}, \qquad \tan 45^\circ = 1$$

$$\sin 30^\circ = \frac{1}{2}, \qquad \cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \qquad \cos 60^\circ = \frac{1}{2}, \qquad \tan 60^\circ = \sqrt{3}$$

Common trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$2\sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2\cos A \cos B = \cos(A - B) + \cos(A + B)$$

$$2\sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$\sin^2 A + \cos^2 A = 1$$

$$1 + \cot^2 A = \text{cosec}^2 A, \quad \tan^2 A + 1 = \sec^2 A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\sin 2A = 2\sin A \cos A$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}, \qquad \cos^2 A = \frac{1 + \cos 2A}{2}$$

$\sin^2 A$ is the notation used for $(\sin A)^2$. Similarly $\cos^2 A$ means $(\cos A)^2$ etc. This notation is used with trigonometric and hyperbolic functions but with positive integer powers only.

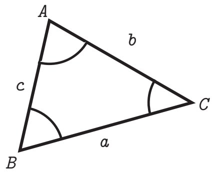
The sine rule and cosine rule

The sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic identities

$$e^x = \cosh x + \sinh x, \quad e^{-x} = \cosh x - \sinh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \text{sech}^2 x$$

$$\coth^2 x - 1 = \text{cosech}^2 x$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

Inverse hyperbolic functions

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \quad \text{for } x \geqslant 1$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad \text{for } -1 < x < 1$$

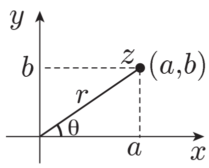
Differentiation

$y = f(x)$	$\frac{dy}{dx} = f\ (x)$
k	

Integration

$f(x)$	$\int f(x)\,dx = F(x) + c$	
k, constant	$kx + c$	
x^n, $(n = -1)$	$\frac{x^{n+1}}{n+1} + c$	
$x^{-1} = \frac{1}{x}$	$\ln x + c \quad x > 0$ $\ln(-x) + c \quad x < 0$	
e^x	$e^x + c$	
$\cos x$	$\sin x + c$	
$\sin x$	$-\cos x + c$	
$\tan x$	$\ln(\sec x) + c$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\sec x$	$\ln(\sec x + \tan x) + c$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\operatorname{cosec} x$	$\ln(\operatorname{cosec} x -$	

Complex Numbers



Cartesian form: $z = a + bj$ where $j = \sqrt{-1}$

Polar form:

$z = r(\cos\theta + j\sin\theta) = r\angle\theta$

$a = r\cos\theta$, $b = r\sin\theta$, $\tan\theta = \frac{b}{a}$

Exponential form: $z = re^{j\theta}$

Euler's relations

$e^{j\theta} = \cos\theta + j\sin\theta$, $\quad e^{-j\theta} = \cos\theta - j\sin\theta$

Multiplication and division in polar form

$$z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2), \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

If $z = r\angle\theta$, then $z^n = r^n\angle(n\theta)$

De Moivre's theorem

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$$

Relationship between hyperbolic and trig functions

$\cos jx = \cosh x$, $\quad \sin jx = j\sinh x$

$\cosh jx = \cos x$, $\quad \sinh jx = j\sin x$

i rather than j may be used to denote $\sqrt{-1}$.

Vectors

If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$

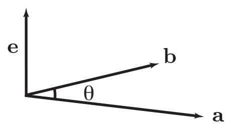
Scalar product

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}\cdot\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Vector product



$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{e}}$

$\hat{\mathbf{e}}$ is a unit vector perpendicular to the plane containing **a** and **b** in a sense defined by the right hand screw rule.

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}\times\mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

a

=

Sequences and Series

Arithmetic progression: $a, a + d, a + 2d, \ldots$

a = first term, d = common difference,

kth term = $a + (k - 1)d$

Sum of n terms, $S_n = \frac{n}{2}(2a + (n - 1)d)$

Sum of the first n integers,

$1 + 2 + 3 + \ldots + n =$

$$\sum_{k=1}^{n} k = \frac{1}{2}n$$

Matrices and Determinants

The 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has determinant

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ has determinant

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(expanded along the first row).

The inverse of a 2×2 **matrix**

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

provided that $ad - bc = 0$.

Matrix multiplication: for 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\beta & a\gamma + b\delta \\ c\alpha + d\beta & c\gamma + d\delta \end{pmatrix}$$

Remember that $AB = BA$ except in special cases.

The Binomial Coefficients

The coefficient of x^k in the binomial expansion of $(1 + x)^n$ when n is a positive integer is denoted by $\binom{n}{k}$ or ${}^nC_k$.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

$0! = 1, \quad n! = n(n-1)!$

so, for example, $4! = 1.2.3.4$

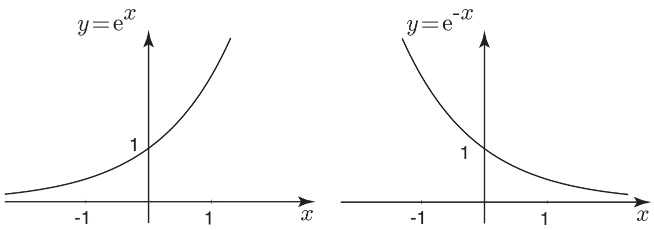
The pattern of the coefficients is seen in

Pascal's triangle:

```
            1       1
        1       2       1
    1       3       3       1
1       4       6       4       1
1   5       10      10      5   1
⋮   ⋮   ⋮   ⋮   ⋮   ⋮   ⋮
```

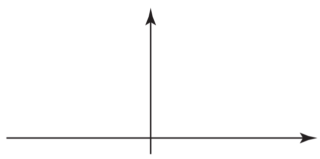
${}^nC_k$ is the number of subsets with k elements that can be chosen from a set with n elements.

Exponential functions



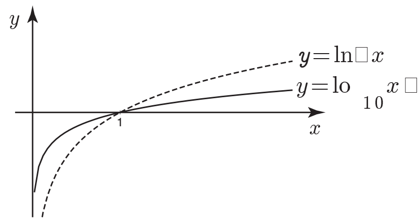
Graph of $y = e^x$ showing exponential growth

Graph of $y = e^{-x}$ showing exponential decay



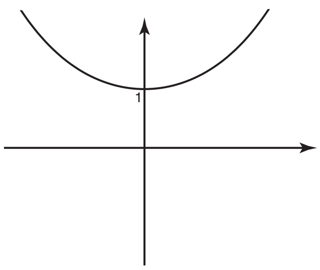
Graphs of $y = 0.5^x$, $y = 3^x$, and $y = 2^x$

Logarithmic functions



Graphs of $y = \ln x$ and $y = \log_{10} x$

Hyperbolic functions



Graphs of $y = \sinh x$, $y = \cosh x$ and $y = \tanh x$